Problem Set I: due Monday, January 25

1. Kulsrud; Chapter 3, #1 — simple, but educational. Highlights duality of lines and fluid elements.

2. Kulsrud; Chapter 3, #3 — introduction to helicity

3. Kulsrud; Chapter 3, #4 — simple illustration of magnetic braking. Instructive.

4. Kulsrud; Chapter 3, #6

5. *Electron MHD* (EMHD)

This extended problem introduces you to EMHD and challenges you to apply what you've learned about MHD to understand the structures of a different system of fluid equations. In EMHD, the ions are stationary and the "fluid" is a fluid of electrons. EMHD is useful in problems involving fast Z-pinches, filamentation and magnetic field generation in laser plasmas, Fast Igniter, etc.

The basic equations of EMHD are the electron momentum balance equation

(1) $$\frac{\partial}{\partial t}\underline{\mathrm{v}} + \underline{\mathrm{v}} \cdot \underline{\nabla}\underline{\mathrm{v}} = -\frac{q}{m}\underline{E} - \frac{\nabla P}{\rho} - \frac{q}{mc}\left(\underline{\mathrm{v}} \times \underline{B}\right) - \nu\underline{\mathrm{v}},$$

(2) $$\underline{J} = -nq\underline{\mathrm{v}},$$

and continuity

(3) $$\underline{\nabla} \cdot \underline{J} = 0.$$

Note that here, Ampere's law forces incompressibility of the mass flow $\rho\underline{\mathrm{v}}$. Here $\underline{\mathrm{v}}$ is the electron fluid velocity, ν is the electron-ion collision frequency, $q = |e|$, $m = m_e$. Of course, Maxwell's equations apply, but the displacement current is neglected.

i.) *Freezing-in*

Determine the freezing-in law for EMHD by taking the curl of Eqn. (1) and using the identity

$$-\underline{v}\cdot\underline{\nabla}\underline{v} = \underline{v}\times\underline{\omega} - \underline{\nabla}\left(v^2/2\right).$$

Assume the electrons have $p = p(\rho)$. Approach this problem by trying to derive an equation for "something" which has the structure of the induction equation in MHD. Discuss the physics - what is the "something" and what is it frozen into? In retrospect, why is the frozen-in quantity obvious? How is freezing-in broken?

ii.) *Large Scale Limit*

Show that for $\ell^2 >> c^2/\omega_{pe}^2$, the dynamical equations for EMHD reduce to

$$\frac{\partial B}{\partial t} + \underline{\nabla}\times\left(\frac{\underline{J}}{nq}\times\underline{B}\right) = -\nu\underline{\nabla}\times\left(\frac{\underline{J}}{nq}\right)$$

$$\underline{\nabla}\cdot\underline{J} = 0\,; \quad \underline{\nabla}\cdot\underline{B} = 0\,.$$

a) Show that density remains constant here.

b) Formulate an energy theorem for EMHD in this limit, by considering the energy content of a "blob" of EMHD fluid.

c) Discuss the frozen-in law in this limit.

6. Consider a magnetic flux tube frozen into a moving fluid.

a) If c_1 and c_2 are any two curves encircling the flux tube, show

$$\oint_{c_1} \underline{A} \cdot d\ell = \oint_{c_2} A \cdot d\ell .$$

b) Show that the strength of the flux tube is constant in time. 'Strength' is defined by the integral in part a.).